Homotopy Types of Schemes

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October 15, 2025

What are the possible homotopy types given by schemes? This question seems to be non-trivial. Here is a post I made on stack exchange.

1 Contractable Schemes

1.1 Affine

Let R be a commutative unital *integral domain*. In particular (0) is a prime ideal and therefore $\mathrm{Spec}(R)$ contains a generic point. is this iff? We claim that it is a contractable space. What we need to produce is a pair of maps

$$* \leftrightarrows \mathrm{Spec}(R)$$

such that the compositions are homotopic to the identity of the respective spaces. There is a unique map to the point and moreover the composition will always be homotopic to the identity, so all we need is a map

$$* \to \mathrm{Spec}(R)$$

such that $\mathrm{Spec}(R) \to * \to \mathrm{Spec}(R)$ is homotopic to the identity. This map is the inclusion of the generic point, denote it (0). Therefore we are claiming that the map

$$\mathrm{Spec}(R) \to \mathrm{Spec}(R)$$

$$\mathfrak{p} \mapsto (0)$$

is homotopic to the identity. Consider the following homotopy

$$H : \mathrm{Spec}(R) \times I \to \mathrm{Spec}(R)$$

$$H(\mathfrak{p}, t) = \begin{cases} \mathfrak{p}, & t = 0 \\ (0), & t > 0 \end{cases}$$

Then it is clear that the two terminal $t = 0, 1$ functions are the identity and the constant map respectively. Using the notation of Hartshorne we consider an open subset of the codomain $\mathrm{Spec}(R) - V(\mathfrak{a})$ whose preimage will therefore be (the second summand is where we use that (0) is generic)

$$\big(\mathrm{Spec}(R) - V(\mathfrak{a})\big) \times \{0\} \cup \big(\mathrm{Spec}(R) \times (0, 1]\big) = \big(\mathrm{Spec}(R) - V(\mathfrak{a})\big) \times I \cup \big(\mathrm{Spec}(R) \times (0, 1]\big)$$

which is the union of two open sets, note that the two sets are open in the product because we are taking a finite product and therefore opens are generated by products of opens, and moreover in the subspace topology $(0, 1]$ is open. Thus the preimage of an open is open, this homotopy is a continuous map and so we are done.

Remark. To be really clear where we are using the generic point, lets look at the more general setting

$$H : X \times I \to X$$

$$H(x, t) = \begin{cases} x, & t = 0 \\ * & t > 0 \end{cases}$$

and let $U \subseteq X$, then the preimage of U is

$$U \times \{0\} \cup \delta_{* \in U} X \times (0, 1]$$

where $\delta_{* \in U}$ indicates that this term is there iff $* \in U$. Now if $* \in U$ we have seen that we can rewrite this as a union of opens, however if it is not then we are left with $U \times \{0\}$ which is not open. Because we let $*$ be the generic point the condition is always satisfied.

Remark. Someone here states that it is sufficient to be irreducible. This seems reasonable but not immediate.

1.2 Globally

Some guy in my stackexchange post claims that this same homotopy above will work for any integral scheme (locally spec of an integral ring) because they have a global generic point. The open sets on such a scheme will be unions of local opens of the form above so the result should go through in the same way, our preimage is a union of things such as above.

2 Non-Contractable Schemes

Discussion of when things are not contractable. Gives a nice example of a space that is weakly homotopic to the circle.

References